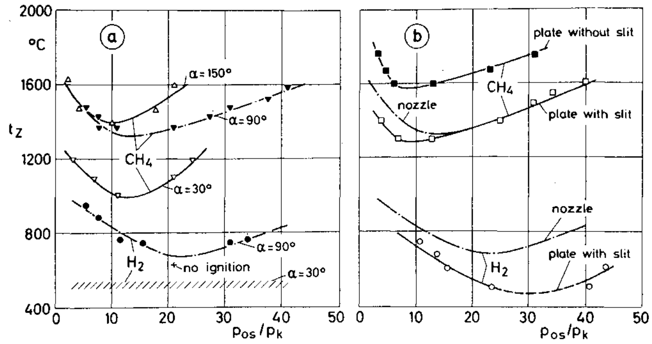

Fig. 2 Ignition temperature of a Mach 2 air stream for fuel injection through nozzles and b) plates.

Fuel Injection through Single Nozzles

Figure 2a shows typical results for fuel gas injection through single nozzles with different inclination angles into an air stream of $M = 2$. The curves for $\alpha = 90°$ were taken from the previous work.[1] Whereas the initial decrease of the ignition temperature with increasing fuel pressure ratio can be explained by the steepening of the bow shock in front of the injected fuel gas, the increase of the ignition temperature beyond the minimum is most probably due to the fuel jet penetrating through the relatively thin air jet of ≈ 10 mm diam.

When the fuel jet is directed *upstream* ($\alpha = 30°$) the ignition temperature decreases markedly compared with vertical injection, the difference between the minimum ignition temperatures for $\alpha = 90°$ and $\alpha = 30°$ amounting to $\approx 400°C$ for methane. With hydrogen injected at an angle $\alpha = 30°$, stable combustion was observed at static air temperatures as low as 500°C for all pressure ratios of the fuel. It was not possible to determine the corresponding minimum ignition temperature as a function of the pressure ratio because, with the present set-up of the arc heater, the static temperature of the Mach 2 air stream could not be reduced below 500°C. On the other hand, the ignition temperature is shifted to higher values, when the fuel gas nozzle is directed *downstream*, as is shown by the curve for methane at $\alpha = 150°$. The influence of the injection angle can easily be explained by the fact that the strength of the bow shock in front of the fuel gas and, therefore, the increase in temperature and density are smaller for downstream than for upstream injection. It is to be expected that for injection angles approaching $\alpha = 180°$ the ignition temperature rises towards the value for parallel fuel admixture.

Fuel Injection through the Plate

Since it was an open question, whether the results obtained for fuel injection through single nozzles into free supersonic air streams can be applied to channel flows, the influence of adjacent walls was investigated qualitatively by injecting the fuel through the plate arrangement shown in Fig. 1b. In one set of the experiments with methane the slit behind the injection port hole was closed (full squares in Fig. 2b). In this case the ignition conditions are worse than for transverse nozzle injection (dash-dotted curve in Fig. 2b). Most probably this result is due to the fact that entrainment of ambient air into the wake downstream of the fuel jet is prevented by the plate. Hence, there recirculates a mixture that is richer in methane and has higher ignition temperatures and larger ignition delay times than for nozzle injection.

Therefore, it was to be expected that the ignition conditions for the plate could be improved by introducing air into the wake, thereby shifting the methane air ratio towards leaner values. In fact, experiments in which the slit was open, yielded essentially lower ignition temperatures: for methane the ignition temperature dropped by $\approx 300°C$ (hollow squares in Fig. 2b). For hydrogen (hollow circles) the ignition temperature was lower than for nozzle injection at $\alpha = 90°$ (dash-dotted curve) over the whole range of pressure ratios. As the static air stream temperature could not be reduced below 500°C, the minimum ignition temperature for hydrogen was estimated to be about 470°C by extrapolating the two branches of the curve.

These experiments show that the ignition conditions can be improved considerably when the fuel gas is injected through a hole in a wall, provided that entrainment of ambient air into the wake downstream of the injection port hole is made possible. Even lower ignition temperatures should be reached by injecting the fuel through a boring in the wall which is inclined upstream to the air flow.

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Swirling Nozzle Flow Equations from Crocco's Relation

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Introduction

SWIRLING nozzle flow is not simple to analyze because it varies radially as well as axially. For example, controversies[1–5] on the axial velocity at the choking throat have not yet been settled because the exact flow equations were not used. However, Lewellen et al.[6] have determined the exact governing equations for flow varying only in the radial direction, i.e., in a constant area region. The purpose of this Note is to show that the swirling phenomena of an inviscid nonisoenergetic and nonisentropic flow are essentially governed by Crocco's relation. It is shown here that Lewellen's result can be obtained directly from this relation. Solutions for some simple swirling flows are given below.

Crocco's Equation

For a steady flow Crocco's equation relates the variation of entropy, vorticity, and total enthalpy in the following manner:

$$T\nabla S + \mathbf{V} \times \mathbf{\Omega} = \nabla h_0 \qquad (1)$$

where $T,S,\mathbf{V},\mathbf{\Omega}$ and h_0 denote, respectively, the temperature, entropy, velocity, vorticity, and total enthalpy of the fluid. For an axisymmetrical flow it is convenient to use cylindrical coordinates (r,θ,z). In this case the radial variation of Crocco's equation appears as

$$T\,\partial S/\partial r + v\Omega_z + w\,\partial w/\partial r = \partial h_0/\partial r \qquad (2)$$

where v and w represent, respectively, the tangential and axial velocity components. The axial derivative of radial velocity was neglected since the nozzle cross-sectional area was assumed to vary slowly. For an axisymmetrical flow the vorticity component in the axial direction may be related to the

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circulation by

$$\Omega_z = v/r + \partial v/\partial r = 1/r(\partial\Gamma/\partial r) \quad (3)$$

where $\Gamma = vr$ is the value of circulation per unit radian. Defining the swirling parameter[7] in the usual manner gives

$$\alpha = [(\gamma - 1)/2]^{1/2}\, \Gamma/a_0R(z) \quad (4)$$

where γ, a_0 and $R(z)$ are, respectively, the ratio of specific heats, stagnation speed of sound, and the nozzle radius. The radial variation of α is therefore given by

$$\partial\alpha/\alpha\partial r = \partial\Gamma/\Gamma\partial r - \partial a_0/a_0\partial r \quad (5)$$

The entropy change in the radial direction due to the variation of stagnation pressure p_0 and stagnation temperature T_0 may be expressed as

$$\partial S/R'\partial r = -\partial p_0/p_0\partial r + [\gamma/(\gamma - 1)]\partial T_0/T_0\partial r \quad (6)$$

where R' is the gas constant. Using Eqs. (3, 5 and 6) together with $h_0 = c_pT_0 = a_0^2/(\gamma - 1)$, Eq. (2) becomes

$$-\frac{a^2}{\gamma p_0}\frac{\partial p_0}{\partial r} + \frac{v^2}{\alpha}\frac{\partial\alpha}{\partial r} + w\frac{\partial w}{\partial r} + \left(\frac{a^2}{\gamma - 1} + \frac{v^2}{2} - \frac{a_0^2}{\gamma - 1}\right)\frac{2}{a_0}\frac{\partial a_0}{\partial r} = 0 \quad (7)$$

where $a^2 = \gamma R'T$ is the local speed of sound.

Radial Equation for Swirling Flow

Using the energy relation along a streamline

$$a^2/(\gamma - 1) + v^2/2 + w^2/2 = a_0^2/(\gamma - 1) \quad (8)$$

and the definition of Mach number

$$M_a = w/a \quad (9a)$$

$$M_t = v/a \quad (9b)$$

Eq. (7) simply reduces to

$$\frac{\partial p_0}{p_0\partial r} + \frac{\gamma M_a^2}{a_0}\frac{\partial a_0}{\partial r} - \frac{\gamma M_t^2}{\alpha}\frac{\partial\alpha}{\partial r} - \frac{\gamma M_a^2}{w}\frac{\partial w}{\partial r} = 0 \quad (10)$$

Differentiation of Eq. (9a) yields

$$(1/w)\partial w/\partial r = (1/M_a)\partial M_a/\partial r + (1/a)\partial a/\partial r \quad (11)$$

and the energy Eq. (8) may be rearranged in terms of Mach numbers as

$$a_0^2 = a^2[1 + (\gamma - 1)(M_t^2 + M_a^2)/2] \quad (12)$$

Again replacing a_0^2 by Eq. (4) and using $\Gamma = vr$, Eq. (12) yields the Mach number relationship

$$M_t^2 = \frac{2}{\gamma - 1}\frac{\alpha^2R^2}{r^2 - \alpha^2R^2}\left(1 + \frac{\gamma - 1}{2}M_a^2\right) \quad (13)$$

Now substituting Eq. (11) and the differential forms of Eqs. (12) and (13) into Eq. (10), Crocco's relation assumes the following form after some manipulation:

$$\frac{\gamma M_a}{1 + \frac{\gamma - 1}{2}M_a^2}\frac{\partial M_a}{\partial r} = \frac{1}{p_0}\frac{\partial p_0}{\partial r} - \frac{2\gamma\alpha R^2}{(\gamma - 1)(r^2 - \alpha^2R^2)} \times \frac{\partial\alpha}{\partial r} - \frac{\gamma\alpha^2R^2M_a^2}{r(r^2 - \alpha^2R^2)} \quad (14)$$

This is the equation first obtained by Lewellen et al. Although $\partial a_0/\partial r$ cancels out in Eq. (14), the effect is hidden in the terms of $\partial p_0/\partial r$ and $\partial\alpha/\partial r$. Equation (14) is applicable to any inviscid nonisoenergetic and nonisentropic, compressible swirling flow as shown in the above derivation. Equation (14) should be solved simultaneously with the axial differential form of Crocco's equation. Since the axial differential form of Crocco's equation has not been obtained in this Note, Eq. (14) is considered to apply to a swirling flow occurring in a constant-area nozzle. For a given distribution of $p_0(r)$ and $\alpha(r)$ in this nozzle, Eq. (14) may be integrated with respect to r to yield the axial Mach number distribution.

Simple Swirling Flow Solutions

Solutions for some simple swirling flows are obtained and discussed below:

a) *Irrotational circulating flow*

In this case, a_0, p_0, and therefore S are constants throughout the entire nozzle, whereas α is constant across a section but will vary with $R(z)$. This flow was treated by Mager[7] about 10 years ago. Mager employed a nondimensional velocity, $M = w/[a_0^2 - \gamma - 1)w^2/2]^{1/2}$, which is not a Mach number as he claimed. Note that w or M is uniform across a nozzle section as seen in Eq. 10), excluding the void region near the axis. The advantage of using this constant property M is to facilitate the mass flow calculation. However, the concept of compressibility (or Mach number) is missing in his work. With p_0 and α being constants, Eq. (14) reduces to

$$\partial M_a/\partial r = -\alpha^2R^2M_a[1 + (\gamma - 1)M_a^2/2]/r(r^2 - \alpha^2R^2) \quad (15)$$

which may be arranged in the following form:

$$\frac{\partial \ln\{M_a^2/[1 + (\gamma - 1)M_a^2/2]\}}{\partial r} = \frac{-2\alpha^2R^2}{r(r^2 - \alpha^2R^2)} \quad (16)$$

Direct integration of Eq. (16) yields the closed form solution

$$M_a^2/[1 + \frac{\gamma - 1}{2}M_a^2] = C_1(z)r^2/(r^2 - \alpha^2R^2) \quad (17)$$

or

$$M_a^2 = C_1(z)/[1 - \frac{\alpha^2R^2}{r^2} - \frac{\gamma - 1}{2}C_1(z)] \quad (18)$$

where $C_1(z)$ is an arbitrary function of z. $C_1(z)$ must be determined from the partial derivative of Crocco's relation with respect to z. For a constant area nozzle $C_1(z)$ is a constant which may be determined from the mass flow relationship or from a given value of w. Note that for $r = \alpha R, M_a^2 = -2/(\gamma - 1) < 0$ which is impossible for this particular example. Recall the definition of α in Eq. (4), which may be expressed as

$$\alpha = r_0v_{\max}/RV_{\max} \quad (19)$$

where r_0 is the radius of void flow region and $V_{\max}^2 = (v^2)_{\max} + w^2 = 2a_0^2/(\gamma - 1)$. Since $v_{\max}$ (at r_0) $< V_{\max}$ in this example, therefore $\alpha R < r_0$. In other words, the fluid is confined only to a region of $r_0 \leq r \leq R$.

b) *Solid body rotation*

If the fluid rotates as a solid body, $\Omega_z = 2\omega = d\Gamma/rdr$, where ω is the constant angular velocity about the z-axis. Substituting this relation in Eqs. (4) or (5) gives

$$\alpha = [(\gamma - 1)/2]^{1/2}\, \omega r^2/a_0R \quad (20a)$$

or

$$\partial\alpha/\partial r = 2\alpha/r \quad (20b)$$

for an isoenergetic flow. Using Eq. (20b), Eq. (14) reduces to

$$\frac{M_a}{[1 + (\gamma - 1)M_a^2/2][1 + (\gamma - 1)M_a^2/4]}\frac{\partial M_a}{\partial r} = \frac{-4\alpha^2R^2}{(\gamma - 1)(r^2 - \alpha^2R^2)r} \quad (21)$$

if p_0 is also constant everywhere. In other words, the flow has to be isentropic everywhere. Substituting Eq. (20a) into Eq. (21), it can be arranged as

$$\frac{\partial \, ln\{[1 + (\gamma - 1)M_a^2/2]/[1 + (\gamma - 1)M_a^2/4]\}}{\partial r} = -(\gamma - 1)(\omega^2 r/a_0^2)/[1 - (\gamma - 1)(\omega^2 r^2/2a_0^2)] \quad (22)$$

Direct integration of Eq. (22) also yields a closed form solution

$$\frac{[1 + (\gamma - 1)M_a^2/2]}{[1 + (\gamma - 1)M_a^2/4]} = C_2(z)[1 - (\gamma - 1)\omega^2 r^2/2a_0^2] \quad (23)$$

where $C_2(z)$ is an arbitrary function of z. $C_2(z)$ must be determined from the partial derivative of Crocco's equation with respect to z. For a constant area nozzle $C_2(z)$ is a constant which may be determined from the mass flow relationship.

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Wall Curvature and Transition Effects in Turbulent Boundary Layers

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Introduction

THIS Note presents one approach by which the eddy viscosity and mixing-length concepts which are being used in the current turbulent boundary-layer prediction methods can be modified to account for streamwise wall curvature and transition effects. A comparison of several calculated results using these modifications in the prediction method of [Ref. 1] show good agreement with experiment.

Analysis

We consider the momentum and energy equations for two-dimensional compressible boundary layers

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = u_e(du_e/dx) + (\partial \tau/\partial y) \quad (1)$$

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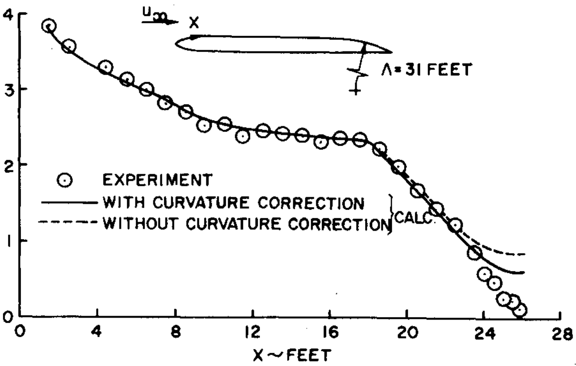

Fig. 1 Comparison of calculated experimental skin-friction coefficients for the data of Schubauer and Klebanoff.

$$\rho u(\partial H/\partial x) + \rho v(\partial H/\partial y) = (\partial/\partial y)(u\tau + q) \quad (2)$$

and the eddy viscosity formulation of Ref. 2, which for inner and outer regions of the boundary layer is defined by ϵ_i and ϵ_0, respectively

$$\epsilon = \begin{cases} \epsilon_i = (ky)^2(1 - \exp(-y/A))^2|\partial u/\partial y| \\ \epsilon_0 = 0.0168\left|\int_0^\infty (u_e - u)dy\right|\left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right]^{-1} \end{cases} \quad (3)$$

In Eq. (3), A is a damping-length constant given by

$$A = A^+(\tau_w/\rho_w)^{-1/2}(\rho/\rho_w)^{1/2}(1/N) \quad (4)$$

where

$$N \equiv \{(\mu/\mu_e)(\rho_e/\rho_w)^2 \, p^+/v_w^+ \, [1 - \exp(11.8\,(\mu_w/\mu)v_w^+)] + \exp(11.8\,(\mu_w/\mu)v_w^+\})^{1/4} \quad (5)$$

$$p^+ \equiv \frac{\nu}{u_e^2}\frac{du_e}{dx}\left(\frac{c_f}{2}\right)^{-3/2}, \quad v_w^+ \equiv v_w\left(\frac{\tau_w}{\rho_w}\right)^{-1/2} \quad (6)$$

The empirical constants k and A^+, which are generally assumed to be constants, vary at low Reynolds numbers[3]

$$k = 0.40 + 0.19/(1 + 0.49Z^2), \quad A^+ = 26 + [14/(1 + Z^2)], \quad Z \geq 0.3 \quad (7)$$

where $Z = R_\theta \times 10^{-3}$.

In Eqs. (1) and (2), τ and q denote the total shear stress and heat-transfer rate, respectively. They are given by

$$\tau = \mu(\partial u/\partial y) - \rho\langle u'v'\rangle, \quad q = k(\partial T/\partial y) - \rho\langle v'H'\rangle \quad (8)$$

which, by the eddy viscosity and turbulent Prandtl number (Pr_t) concepts, can also be written as (see Ref. 1)

$$\tau = \mu\frac{\partial u}{\partial y} + \rho\epsilon\frac{\partial u}{\partial y}, \; q = k\frac{\partial T}{\partial y} + \rho\frac{\epsilon}{Pr_t}\frac{\partial H}{\partial y} \quad (9)$$

The eddy viscosity expressions in the form given in Eq. (3) account for pressure gradient and heat and mass transfer effects quite well. They can be generalized to account for streamwise wall curvature effect by multiplying them (both inner and outer eddy viscosity expressions) by S^2, an ex-

Table 1 Extent of transitional Reynolds number at two blade Reynolds numbers

$Re_b \times 10^{-5}$	$Re_{tr} \times 10^{-5}$	$Re_{\Delta x} \times 10^{-5}$
1	0.1	0.28
1	0.5	0.81
10	1.0	1.30
10	5.0	3.80